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THE MINIMAL NORMAL EXTENSION FOR M_z ON THE HARDY SPACE OF A PLANAR REGION

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ABSTRACT. Multiplication by the independent variable on $H^2(R)$ for R a bounded open region in the complex plane $\mathbb{C}$ is a subnormal operator. This paper characterizes its minimal normal extension N. Any normal operator is determined by a scalar-valued spectral measure and a multiplicity function. It is a consequence of some standard operator theory that a scalar-valued spectral measure for N is harmonic measure for R, ω. This paper investigates the multiplicity function m for N. It is shown that m is bounded above by two ω-a.e., and necessary and sufficient conditions are given for m to attain this upper bound on a set of positive harmonic measure. Examples are given which indicate the relationship between N and the boundary of R.

1. INTRODUCTION AND PRELIMINARIES

One of the major difficulties encountered in the study of subnormal operators is the shortage of examples. This paper examines the subnormal operator defined by multiplication by the independent variable on $H^2(R)$, the space of analytic functions f on a bounded region R such that $|f|^2$ has a harmonic majorant. Recall that there exists a finite Borel measure ω_a carried by ∂R with the property that for all $f \in C_{\mathbb{R}}(\partial R)$, $\hat{f}(a) = \int_{\partial R} \hat{f}\, d\omega_a$ where $\hat{f}$ denotes the solution of the Dirichlet problem for R associated with f. This measure is known as *harmonic measure* for R at a. For more on this see [11, p. 332].

The following result is probably known, but seems to be documented only for the case of Dirichlet regions. Let σ be a normalized arc-length measure for the unit disk D.

1.1. **Theorem.** *Let R be a bounded region and let $\phi: D \to R$ be a uniformization map from D onto R which takes 0 to a. Let $\tilde{\phi}$ be a Borel function on ∂D which agrees σ-a.e. with the radial limits of ϕ. Then $\omega_a(E) = \sigma \circ \tilde{\phi}^{-1}(E)$ for any Borel set E.*

Proof. For f in $C_{\mathbb{R}}(\partial R)$, define $\hat{f}$ by $\hat{f}(x) = \int_{\partial R} f\, d\omega_x$, where $\hat{f}$ is a bounded harmonic function on R. Let $A \subset \partial D$ have the following properties: (i) $\sigma(A) = 1$, (ii) the radial limits of ϕ and $\hat{f} \circ \phi$ exist on A, and (iii)

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the radial limits of ϕ agree with $\tilde{\phi}$ on A. The radial limit function associated with $\hat{f}\circ\phi$ will be denoted by $\widetilde{\hat{f}\circ\phi}$. Choose $\lambda\in A$ such that $\tilde{\phi}(\lambda)$ is a regular boundary point. Then as $r\to 1$, we have that $\phi(r\lambda)\to\tilde{\phi}(\lambda)$, which implies that $\hat{f}(\phi(r\lambda))\to f(\tilde{\phi}(\lambda))$. By [27, p. 42] the set I of irregular points is a countable union of compact sets of capacity zero; and, by [14, p. 119] $\tilde{\phi}^{-1}(I)$ has measure zero. Thus, we have that $\widetilde{\hat{f}(\phi)} = f\circ\tilde{\phi}$, which holds σ-a.e. This allows us to define a bounded harmonic function g on D by

$$g(\alpha) = \int_{\partial D} f\circ\tilde{\phi}(\lambda)\, d\tau_{\alpha(\lambda)},$$

where τ_α is harmonic measure for D at α. Since the radial limits of g and $\hat{f}\circ\phi$ agree, then $g = f\circ\phi$ and so $g(0) = \hat{f}(\phi(0)) = \hat{f}(a)$. Thus,

$$\int_{\partial R} f\, d\omega_a = \int_{\partial D} f\circ\tilde{\phi}(\lambda)\, d\sigma(\lambda). \quad \square$$

Hereafter let D be a region defined by the removal from the open unit disk of the closure of a countable pairwise disjoint collection of closed subdisks. Furthermore, we will assume that the set of accumulation points of the subdisks, those points with the property that every neighborhood of them contains one of the subdisks, is countable and that 0 lies in D. Let R be a bounded region in $\mathbb{C}$ which is conformally equivalent to D. If R is simply connected, let D be the unit disk. Let $\phi\colon D\to R$ be a one-to-one conformal map such that $a=\phi(0)$. Harmonic measure for D at 0 will be denoted by τ_0 and ω_a will denote harmonic measure for R at a.

The operator obtained by composition with ϕ is an isomorphism between $H^2(R)$ and $H^2(D)$, and $H^2(D)$ embeds onto a subspace $H^2(\partial D)$ of $L^2(\partial D, \tau_0)$ via nontangential limits. Let $\tilde{\phi}$ be a Borel function on ∂D equal τ_0-a.e. to the nontangential limits of ϕ. It is easy to show that $\omega_a = \tau_0\circ\tilde{\phi}^{-1}$. Hereafter S will denote the operator on $H^2(R)$ defined by multiplication by the independent variable. S is unitarily equivalent to M_ϕ on $H^2(D)$ which has a normal extension, namely $M_{\tilde{\phi}}$ on $L^2(\partial D, \tau_0)$. Thus, S is subnormal. In this paper we will characterize its minimal normal extension, denoted by N, and study the relationship between N and R. To do this we will make use of the fact that every normal operator is characterized by a scalar-valued spectral measure and a multiplicity function. It will be shown later (Theorem 2.4) that $M_{\tilde{\phi}}$ on $L^2(\partial D, \tau_0)$ is not only a normal extension of S, but, in fact, the minimal one N. Since $\omega_a = \tau_0\circ\tilde{\phi}^{-1}$, Proposition 8.12 in [8, p. 296] shows that harmonic measure for R is a scalar-valued spectral measure for N.

2. The multiplicity function

What is the multiplicity function for N? We will discover that the structure of ∂R determines the answer.

Abrahamse and Kriete have done some pioneering work on multiplicity theory for general multiplication operators. They have established a connection between the multiplicity function m and $\tilde{\phi}^{-1}$. In order to understand this, a definition is required.

2.1. **Definition.** A *disintegration* of τ_0 with respect to $\tilde{\phi}$ is a function $y \to \tau_0^y$ from the essential range of $\tilde{\phi}$ into the set of regular Borel probability measures on D such that

(a) $\tau_0^y(\partial D \backslash \tilde{\phi}^{-1}(y)) = 0$, ω_a-a.e., and

(b) for every Borel set E in ∂D, the function $y \to \tau_0^y(E)$ is ω_a-measurable and $\tau_0(E) = \int_{\partial R} \tau_0^y(E)\, d\omega_a(y)$.

It may be shown (see [1]) that the disintegration is unique and depends only on the equivalence class of $\tilde{\phi}$ in $L^\infty(\partial D, \tau_0)$. The following result, proven in [1, 3], shows the importance of this concept.

2.2. **Theorem.** *Let X be a locally compact separable metric space, and let μ be a Borel probability measure on X. If ψ is in $L^\infty(\mu)$, μ_y is a disintegration of μ with respect to ψ, and m is the multiplicity function for the operator M_ψ on $L^2(\mu)$, then*

$$m(y) = \#\operatorname{supp}(\mu_y) \quad \text{for } \mu\text{-almost every } y$$

and

$$\operatorname{supp}(\mu_y) \subset \operatorname{cl}\{\psi^{-1}(y)\} \quad \text{for } \mu\text{-almost every } y.$$

The difficulty with using this theorem lies in the fact that it is often hard to obtain a disintegration of the measure. It does tell us, however, that $m(y) \leq \#\bar{\psi}^1(y)$, μ-a.e. This can be used to obtain an upper bound on the multiplicity function for N provided that we can show that $N \cong M_{\tilde{\phi}}$ on $L^2(\partial D, \tau_0)$ and provided that we can gain some knowledge of $\tilde{\phi}^{-1}$. To accomplish this we resurrect an old topological result of R. L. Moore [24]. Professor N. K. Nikolskii drew the author's attention to this result. It states that any pairwise disjoint collection of triods (homeomorphs of the letter "T"), in the complex plane can be at most countable. Now let us define a Borel set B contained in ∂R of full ω_a-measure such that B is contained in the set of nontangential limits of ϕ and $\tilde{\phi}$ is equal to the nontangential limit function on $\tilde{\phi}^{-1}(B)$. The argument below is basically Nikolskii's and is a very interesting application of Moore's theorem.

2.3. **Lemma.** $\{x \in B : \tilde{\phi}^{-1}(x) \geq 3\}$ *is countable.*

Proof. Suppose $\#\tilde{\phi}^{-1}(x) \geq 3$. It may be shown via a small topological argument that $\tilde{\phi}^{-1}(x)$ lies in exactly one of the components of ∂D. Since the collection of accumulation points of the subdisks removed in the construction of D is countable, we may consider only those x for which $\tilde{\phi}^{-1}(x)$ contains no such accumulation point. Let $\partial B(z, \delta)$ be the component of ∂D containing $\tilde{\phi}^{-1}(x)$, and choose a_1, a_2, a_3 in $\tilde{\phi}^{-1}(x)$. Assume for the moment that $B(z, \delta)$ is not the unit disk. One can always choose ε_x so that

$$\bigcup_{i=1}^{3} \phi(\{a_i + t(a_i - z) : 0 < t \leq \varepsilon_x\}) \cup \{x\}$$

will form a triod. For $y \neq x$ with $\#\tilde{\phi}^{-1}(y) \geq 3$ and $\tilde{\phi}^{-1}(y) \subset \partial B(z, \delta)$ any corresponding triod would be disjoint from the one above. This is true because ϕ is one-to-one, and these triods are formed by taking the images under ϕ of line segments extending radially from $\partial B(z, \delta)$. Such line segments are, of course, disjoint. Hence it is possible to choose a pairwise disjoint collection of triods which lies in a one-to-one correspondence with the set of points x such that $\#\tilde{\phi}^{-1}(x) \geq 3$ and $\tilde{\phi}^{-1}(x) \subset \partial B(z, \delta)$. By Moore's result this set must be countable. For those x such that $\tilde{\phi}^{-1}(x)$ is contained in the unit circle form triods of the form

$$\bigcup_{i=1}^{3} \phi(\{ta_i : 1 - \varepsilon_x \leq t < 1\}) \cup \{x\}$$

and follow the same argument. Since the number of components of ∂D is countable, the result follows. □

The following theorem will tell us that N, the minimal normal extension of S, actually is $M_{\tilde{\phi}}$ on $L^2(\partial D, \tau)$. The technique used in the proof was discovered by Professor John B. Conway [10].

2.4. **Theorem.** *$N \cong M_{\tilde{\phi}}$ on $L^2(\partial D, \tau_0)$.*

Proof. Let A_0 be the closed linear span of $\{\bar{\tilde{\phi}}^k f : f \in H^2(D),\ k \geq 0\}$ in $L^2(\partial D, \tau_0)$. It must be shown that $A_0 = L^2(\partial D, \tau_0)$. If $h \in L^2(\partial D, \tau_0) \ominus A_0$, then for all $k \geq 0$ and for all f in $H^2(D)$,

$$\begin{aligned} 0 = \langle \bar{\tilde{\phi}}^k f, h \rangle &= \int_{\partial D} \bar{\tilde{\phi}}^k f \bar{h}\, d\tau_0 \\ &= \int_{\partial R} \left[\int_{\tilde{\phi}^{-1}(x)} \bar{\tilde{\phi}}^k f \bar{h}\, d\lambda_x \right] d\omega_a(x) \end{aligned}$$

where ω_a is harmonic measure for R and λ_x is a disintegration of τ_0 with respect to $\tilde{\phi}$. Clearly, f may be replaced by $\tilde{\phi}^j f$ for all $j \geq 0$. Therefore,

$$\begin{aligned} 0 &= \int_{\partial R} \left[\int_{\tilde{\phi}^{-1}(x)} \bar{\tilde{\phi}}^k \tilde{\phi}^j f \bar{h}\, d\lambda_x \right] d\omega_a(x) \\ &= \int_{\partial R} \bar{x}^k x^j \int_{\tilde{\phi}^{-1}(x)} f \bar{h}\, d\lambda_x\, d\omega_a(x). \end{aligned}$$

This tells us that $\int_{\tilde{\phi}^{-1}(x)} f \bar{h}\, d\lambda_x = 0$ for ω_a-a.e. x, by the Weierstrass approximation theorem. Thus for each f in $H^2(R)$ there exists $G_r \subset R$ such that $x \notin G_f$ implies $\int_{\tilde{\phi}^{-1}(x)} f\bar{h}\,d\lambda_x = 0$ and $\omega_a(G_f) = 0$. Let $A = \{x \in \partial R\colon \#\tilde{\phi}^{-1}(x) \le 2\}$. A previous lemma shows us that $\omega_a(A) = 1$. Denote by P the collection of polynomials with complex rational coefficients, and let $G = (\bigcup_{p\in P} G_p) \cup (\partial R\backslash A)$. Clearly $\omega_a(G) = 0$ since P is a countable set. By taking uniform limits, we find that for any polynomial p it follows that $\int_{\tilde{\phi}^{-1}(x)} p\bar{h}\,d\lambda_x = 0$ for $x \in G$. Choose $\alpha \in \tilde{\phi}^{-1}(\partial R\backslash G)$ and $x \notin G$ such that $\alpha \in \tilde{\phi}^{-1}(x)$. There are now two cases to consider.

(1) $\#\tilde{\phi}^{-1}(x) = 1$. Here λ_x is a point mass, and since 1 is a polynomial $0 = \int_\alpha \bar{h}\,d\lambda_x = \bar{h}(\alpha)$.

(2) $\#\tilde{\phi}^{-1}(x) = 2$. Let $\beta \in \tilde{\phi}^{-1}(x)$, $\beta \ne \alpha$. Then $\int_{\tilde{\phi}^{-1}(x)} (z-\beta)h\,d\lambda_x = 0$, which implies that $\bar{h}(\alpha) = 0$ or that $\lambda_x(\alpha) = 0$. Let

$$B = \{\alpha \in \tilde{\phi}^{-1}(\partial R\backslash G)\colon h(\alpha) \ne 0\}.$$

B is contained in

$$\{\alpha \in \tilde{\phi}^{-1}(\partial R/G)\colon \text{ if } \alpha \in \tilde{\phi}^{-1}(x) \text{ then } \lambda_x(\alpha) = 0\}$$

which implies that

$$\tau_0(B) = \int_{\partial D} \chi_B(x)\,d\tau_0(x) = \int_{\partial R}\left[\int_{\tilde{\phi}^{-1}(x)} \chi_B\,d\tau_0\right] d\omega_a(x) = 0$$

since for any $x \notin G$, $\lambda_x(B \cap \tilde{\phi}^{-1}(x)) = 0$ by definition of B. Therefore, τ_0-a.e. $h = 0$. So $A_0 = L^2(\partial D, \tau_0)$. □

Since countable sets have harmonic measure zero, we have a nice upper bound on the multiplicity function.

2.5. **Theorem.** *$m(y) \le 2$ ω_a-a.e.*

Proof. The result follows from the previous theorem, Lemma 2.3, and the fact that $m(y) \le \#\tilde{\phi}^{-1}(y)$ ω_a-a.e. □

The technique in the proof of Theorem 2.5 immediately gives the next result, though there are certainly other methods of arriving at the same conclusion.

2.6. **Theorem.** *If R is bounded by a Jordan curve, then N is of multiplicity one.*

Proof. The hypothesis implies that all the prime ends of R are of the first kind and that every point of ∂R lies in the impression of exactly one prime end. This means that ϕ is continuous to the boundary of D and is one-to-one there. Clearly, $m(y) \le \#\tilde{\phi}^{-1}(y) = 1$ ω_a-a.e. □

Hereafter we will let $B \subset \partial R$ be a Borel set of full ω_a measure contained in the nontangential limits of ϕ such that $\tilde{\phi}$ is equal to the nontangential limit function on $\tilde{\phi}^{-1}(B)$ as before, and

(a) for every $x \in B$, $\#\tilde{\phi}^{-1}(x) \le 2$,

(b) for every $x \in B$, $m(x) = \#\operatorname{supp}(\tau_0^x)$ where τ_0^x is a fixed disintegration of τ_0 with respect to $\tilde{\phi}$, and

(c) $B \subset \operatorname{essran}(\tilde{\phi})$.

Our next goal is to determine whether it is possible for the multiplicity function to attain its upper bound on a set of positive harmonic measure. With this in mind let $A \subset \partial R$ be a Borel set with the following properties:

(1) $\omega_a(A) > 0$,

(2) for every $x \in A$, $\#\tilde{\phi}^{-1}(x) = 2$.

It may be shown using a measurable selection function $s: A \to \tilde{\phi}^{-1}(A)$ [25, p. 515] that there exist disjoint Borel sets $A_1 = s(A)$ and $A_2 = \tilde{\phi}^{-1}(A)\backslash A_1$ contained in ∂D such that for every $x \in A$ one element of $\tilde{\phi}^{-1}(x)$ lies in each A_i. Define Borel measures ω_i, $i = 1, 2$ on A by $\omega_i(E) = \tau_0(A_i \cap \tilde{\phi}^{-1}(E))$, $i = 1, 2$.

2.7. **Theorem.** *The multiplicity function is equal to* 2 *almost everywhere on* A *if and only if* ω_a, ω_1, *and* ω_2 *are pairwise mutually absolutely continuous.*

Proof. Suppose $m = 2$ a.e. on A. Assume that ω_1 is not absolutely continuous with respect to ω_2. Then there exists a Borel set $A' \subset A$ such that $\omega_1(A') \neq 0$ and $\omega_2(A') = 0$. Let us relabel and let $A = A'$, $\omega_i = \omega_i|_{A'}$, $A_1 = s(A')$, and let $A_2 = \tilde{\phi}^{-1}(A')\backslash A_1$. Clearly,

$$N \cong M_{\tilde{\phi}}|_{L^2(\partial D/(A_1 \cup A_2), \tau_0)} \oplus M_{\tilde{\phi}}|_{L^2(A_1, \tau_0)} \oplus M_{\tilde{\phi}}|_{L^2(A_2, \tau_0)}. \tag{2.7.1}$$

Theorem 1.1 tells us that $\tau_0(A_2) = \omega_2(A) = 0$, so

$$N \cong M_{\tilde{\phi}}|_{L^2(\partial D\backslash(A_1 \cup A_2), \tau_0)} \oplus M_{\tilde{\phi}}|_{L^2(A_1, \tau_0)}.$$

Since $\tilde{\phi}$ is one-to-one on A_1 and $m(y) \leq \#\tilde{\phi}^{-1}(y)$ a.e., it is clear that $M_{\tilde{\phi}}|_{L^2(A_1, \tau_0)}$ is a multiplicity one normal operator. Since $\mathrm{supp}(\tau_0^y) \subset \mathrm{cl}(\phi^{-1}(y))$ almost everywhere and $\#\tilde{\phi}^{-1}(y) \leq 2$ a.e. the multiplicity function of $M_{\tilde{\phi}}|_{L^2(\partial D\backslash(A_1 \cup A_2), \tau_0)}$ is zero a.e. on A. This means that for almost all $x \in A$, $m(x) = 1$ contradicting the hypothesis.

Let us assume that ω_1 and ω_2 are m.a.c. Of course, it follows from the fact that $\omega_a|_A = \omega_1 + \omega_2$ that ω_i and ω_a are m.a.c., $i = 1, 2$. Define $k_i(x)$, $i = 1, 2$, by $k_i(x) = (d\omega_a/d\omega_i)^{1/2}(x)$ and define $U: L^2(A, \omega_a)^{(2)} \to L^2(A_1 \cup A_2, \tau_0)$ by

$$U(f_1 \oplus f_2) = \begin{cases} k_1(\tilde{\phi}(x))f_1(\tilde{\phi}(x)), & x \in A_1, \\ k_2(\tilde{\phi}(x))f_2(\tilde{\phi}(x)), & x \in A_2. \end{cases}$$

Note that

$$\begin{aligned}\|U(f_1 \oplus f_2)\|^2 &= \int_{A_1 \cup A_2} |U(f_1 \oplus f_2)|^2 \, d\tau_0 \\ &= \sum_{i=1}^{2} \int_{A_i} (d\omega_a/d\omega_i(\tilde{\phi}(x)))|f_i(\tilde{\phi}(x))|^2 \, d\tau_0(x) \\ &= \sum_{i=1}^{2} \int_{A_i} |f_i(x)|^2 \, d\omega_a(x) \\ &= \|f_1\|^2 + \|f_2\|^2 = \|(f_1 \oplus f_2)\|^2 .\end{aligned}$$

Therefore, U is an isometry. We need to show that it is also surjective and therefore unitary.

First observe that since $\tilde{\phi}$ is one-to-one on A_i, $i = 1, 2$, then $h_i(x) = \tilde{\phi}^{-1}(x) \cap A_i$ defines a function for $i = 1, 2$. Also, a one-to-one Borel image of a Borel set is a Borel set (see [19, p. 489]). This means that $h_i(x)$ is a Borel function. Let $f \in L^2(A_1 \cup A_2, \tau_0)$ be a Borel function. Since $h_i(x)$ is Borel, the following holds:

$$f(x) = U(1/k_1(x))f(h_1(x)) \oplus 1/k_2(x)f(h_2(x)) .$$

Thus U is unitary. Let $T = M_z$ on $L^2(A, \omega_a)$. Then

$$\begin{aligned}UT^{(2)}(f \oplus g) &= U(zf \oplus zg) \\ &= \begin{cases} k_1(\tilde{\phi}(x))\tilde{\phi}(x)f(\tilde{\phi}(x)), & x \in A_1, \\ k_2(\tilde{\phi}(x))\tilde{\phi}(x)g(\tilde{\phi}(x)), & x \in A_2, \end{cases} \\ &= M_{\tilde{\phi}} U(f \oplus g) .\end{aligned}$$

Hence $T^{(2)} \cong M_{\tilde{\phi}}|_{L^2(A_1 \cup A_2, \tau_0)}$. Clearly $T^{(2)}$ is a multiplicity two normal operator. By the decomposition in (2.7.1) we find that

$$N \cong T^{(2)} \oplus M_{\tilde{\phi}}|_{L^2(\partial D \setminus (A_1 \cup A_2), \tau_0)} .$$

Again the second term contributes no multiplicity to A. Therefore the result follows. □

The preceding theorem gives a necessary and sufficient measure-theoretic condition for the multiplicity function to attain its upper bound on a set of positive harmonic measure. This condition may in fact be used in a large number of examples. If R is the slit disk and A is some subinterval of the slit, it is clear that A_1 and A_2 may be taken to be subarcs of ∂D. A well-known result of F. and M. Riesz [30] may be used to show that ω_i and arclength measure are mutually absolutely continuous, for $i = 1, 2$, so m is 2 a.e. on A. This example may be generalized to other simply connected regions which contain a subset V of their boundaries with the property that for all $x \in V$ there exists $\varepsilon_x > 0$

such that

(1) $\overline{B}(x, \varepsilon_x) \cap \partial R$ is a rectifiable Jordan crosscut of $B(x, \varepsilon_x)$ and

(2) $B(x, \varepsilon_x) \backslash \partial R \subset R$.

Such a set V will be called *slitlike.*

The condition in Theorem 2.7 is not always satisfied, however. Consider Figure 1. Begin with an open rectangle slit along the interval from 0 to 1. As in the construction of a Cantor set of positive linear measure, remove a middle interval of length smaller than $\frac{1}{3}$. Above the endpoints of this interval draw two vertical line segments of length $\frac{1}{2}$ starting at a_1 and a_2 as shown. For the time being, connect a_1 to a_2 with a horizontal segment. Let λ_2 be harmonic measure at the point a for the rectangle with vertices a_1, a_2, α_1, and α_2. Construct horizontal segments Γ_1 and Γ_2 of equal length so that

$$\lambda_2\{\overline{\alpha_1\alpha_2} \backslash (\Gamma_1 \cup \Gamma_2)\} < \tfrac{1}{2}.$$

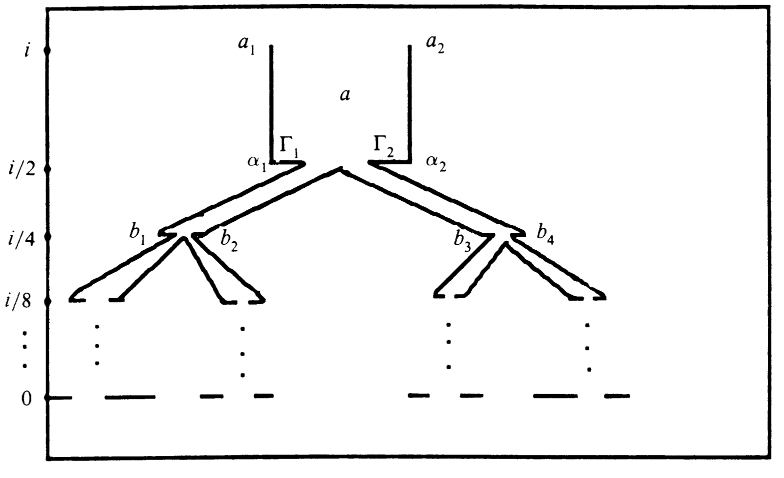

FIGURE 1

Now remove the next two intervals in the construction of the aforementioned Cantor set and draw "branches" as shown. The four points b_1, b_2, b_3, and b_4 illustrated lie above the endpoints of the intervals removed as before. Let λ_3 be harmonic measure at the point a for the region enclosed by drawing line segments from b_1 to b_2 and from b_3 and b_4. Construct horizontal segments Γ'_i, $i = 1, 2, 3, 4$, at the ends of the branches so that

$$\lambda_3\{\overline{b_1b_2} \backslash (\Gamma'_1 \cup \Gamma'_2) \cup \overline{b_3b_4} \backslash (\Gamma'_3 \cup \Gamma'_4)\} < \tfrac{1}{3}.$$

Continue in this manner. The branches will "converge" down to the Cantor set. Call the union of the branches the tree. The tree forms a simply connected domain, since any loop in it being compact must lie above the real axis, and if one slices the tree horizontally anywhere above the real line, then the subregion above the cut is a contractible space. It may also be shown that the tree is a

Jordan domain. This may be accomplished by either showing that all the prime ends are of the first kind and every point on the boundary of the tree lies in the impression of exactly one prime end or by observing that the connection of the boundary of the tree is destroyed by the removal of any two arbitrary points. That the latter statement is equivalent to being a Jordan domain is a result found in [26, p. 93]. Both of these arguments are fairly straightforward and are left to the reader. Let C denote the Cantor set. The harmonic measure of C with respect to the tree is majorized by each of

$$\lambda_2(\overline{\alpha_1\alpha_2}\backslash(\Gamma_1 \cup \Gamma_2)),\ \lambda_3\{\overline{b_1b_2}\backslash(\Gamma_1' \cup \Gamma_2') \cup \overline{b_3b_4}\backslash(\Gamma_3' \cup \Gamma_4')\}, \ldots .$$

Therefore it is zero. Now remove the segment from a_1 to a_2, and let R be the open region illustrated in Figure 1. Since the segment from a_1 to a_2, a subarc of the boundary of the tree, was removed, R is an open rectangle minus two Jordan arcs which intersect only at 0. Thus R is simply connected. Define ϕ, $\tilde{\phi}$, and ω_a as before. By constructing a rectifiable Jordan subdomain of R below the real line whose boundary contains $[0, 1]$ it may be shown that $\omega_a(C) > 0$, because C has positive arc-length measure. The result of F. and M. Riesz in [30] states that arclength and harmonic measure are mutually absolutely continuous for rectifiable Jordan domains. Also for almost every x in C, $\#\tilde{\phi}^{-1}(x) = 2$ since x corresponds to a prime end defined by a chain in the tree and one defined by a chain below the real line. Let A_1 be those points in ∂D corresponding to prime ends of C defined by chains in the tree and let $A_2 = \tilde{\phi}^{-1}(C)\backslash A_1$. These sets are easily seen to be Borel. Define ω_1 and ω_2 as before. Since harmonic measure is carried between Jordan domains such as the tree and $\tilde{\phi}^{-1}$ (tree) by conformal maps, and ω_1 and harmonic measure for the tree are certainly mutually absolutely continuous, $\omega_1(C) = 0$. Since $\omega_a(C) > 0$ and $\omega_a|_C = \omega_1 + \omega_2$, it is clear that $\omega_2(C) > 0$. Therefore, ω_1 and ω_2 are not mutually absolutely continuous; so, there must exist $V \subset C$ of positive harmonic measure such that the multiplicity function is equal to one almost everywhere on V. For another example see [20].

It is well known that a normal operator has multiplicity identically one if and only if it has a star-cyclic vector. This fact along with some of our previous examples may be used to answer some approximation theory questions.

2.8. **Theorem.** *If R is the slit disk, then the polynomials are not dense in $H^2(R)$. In fact, $S = M_z$ on $H^2(R)$ has no cyclic vector.*

Proof. It is clear by the minimality of N that

$$\begin{aligned} L^2(\partial D, \tau) &= \mathrm{cls}\{N^{*n}f : n \geq 0,\ f \in H^2(D)\} \\ &= \mathrm{cls}\{\bar{\tilde{\phi}}^n f : n \geq 0,\ f \in H^2(D)\} \end{aligned}$$

where cls denotes the closed linear span. If the polynomials were dense in $H^2(R)$, then polynomials in ϕ would be dense in $H^2(D)$ since the map T: $H^2(R) \to H^2(D)$ which takes f to $f \circ \phi$ is an isometric isomorphism. This means that

$$L^2(\partial D, \tau) = \text{cls}\{\bar{\tilde{\phi}}^n \tilde{\phi}^m : n, m \geq 0\} = \text{cls}\{R1 : R \in C^*(N)\}.$$

In other words, one is a star-cyclic vector for N. This, of course, cannot be since the multiplicity function is two on the slit. If f is a cyclic vector for S, repeat the above argument for polynomials in $(\phi)(f \circ \phi)$ to obtain a contradiction. □

Again we can generalize this to any region with a set of multiplicity two which has positive harmonic measure. Slitlike sets come to mind for instance. The advantage of this approach lies in the fact that one can tell by inspection that the polynomials are not dense. It is easy to recognize slits.

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